Cube tilings subject to lattice constraints & exponential bases with integer frequencies

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joint work with Andrei Caragea, Dae Gwan Lee, Romanos Malikiosis, Shauna Revay, David Walnut, Romanos Malikiosis

For which parallelipiped $A[0,1]^d \subseteq \mathbb{R}^d$ exists $\Psi \subseteq \mathbb{Z}^d$ so that $\{e^{2\pi i \psi \cdot x}\}_{\psi \in \Psi}$ is an orthogonal basis for $L^2(A[0,1]^d)$?

$$d = 1$$

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Only: $\{e^{2\pi i \psi x}\}_{\psi \in N\mathbb{Z}}$ is an orthogonal basis for $L^2[0, \frac{1}{N}]$, $N \in \mathbb{N}$.

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Theorem (Seip '95)
There exists $\Lambda \subseteq \mathbb{Z}$ with $\mathcal{E}(\Lambda) = \{e^{2\pi i \lambda x}\}_{\lambda \in \Lambda}$ is a Riesz basis for $L^2[0, a]$.

Riesz basis (RB)

$\{g_n\}_{n\in\mathbb{Z}}$ is a RB for H if $\overline{\mathrm{span}\{g_n\}} = H$ and $\exists A, B > 0$: $A\sum_n |a_n|^2 \leq \| \sum_n a_n g_n \|^2 \leq B \sum_n |a_n|^2$, $\forall \{a_n\}$.

Prologue

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Fourier series

$\mathcal{E}(\mathbb{Z}) = \{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is ONB for $L^2[0,1]$ $\quad\Rightarrow\quad$ $\mathcal{E}(\frac{\mathbb{Z}+\alpha}{a}) = \{e^{2\pi i \frac{n+\alpha}{a} x}\}_{n\in\mathbb{Z}}$ is an OB for $L^2[b, b+a]$.

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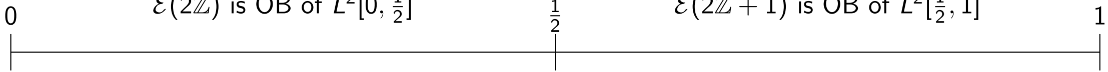
Motivating observation from grad school

$\mathcal{E}(2\mathbb{Z})$ is OB of $L^2[0,\frac{1}{2}]$ $\qquad$ $\mathcal{E}(2\mathbb{Z}+1)$ is OB of $L^2[\frac{1}{2},1]$



$\mathcal{E}(\mathbb{Z}) = \mathcal{E}\big((2\mathbb{Z}+1) \,\dot\cup\, 2\mathbb{Z}\big)$ is ONB of $L^2[0,1]$

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$\mathcal{E}(\Lambda)$ is RB of $L^2[0, a]$

$0 \qquad a \qquad 1$

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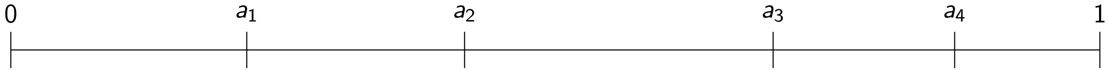
$\mathcal{E}(\Lambda)$ is RB of $L^2[0, a]$ $\qquad$ $\mathcal{E}(\mathbb{Z}\setminus\Lambda)$ is RB of $L^2[a, 1]$

0 $\qquad$ a $\qquad$ 1

Results



Theorem A (GP, Shauna Revay, David Walnut '24)

For $0{=}a_0 < a_1 < \ldots < a_n{=}1$ exists a partition $\Lambda_1,\ldots,\Lambda_n$ of $\mathbb{Z}$ s.t. $\mathcal{E}(\Lambda_k)$ is a Riesz basis of $L^2[a_{k-1},a_k]$.

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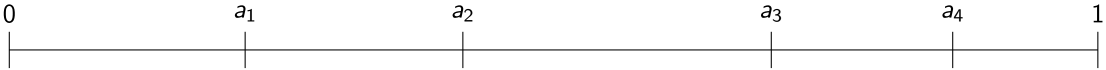

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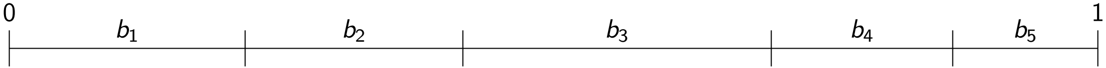

Theorem B (GP, Shauna Revay, David Walnut '24)
For $b_1, \ldots, b_n > 0$ with $\sum_{j=1}^n b_j = 1$ exists a partition Λ_1,...,Λ_n of $\mathbb{Z}$ so that for any $J \subseteq \{1, \ldots, n\}$ we have $\mathcal{E}\big(\bigcup_{j\in J} \Lambda_j\big)$ is a Riesz basis for $L^2(I)$ where I is an **interval** of length $\sum_{j\in J} b_j$.

3 TOOLS

KADEC'S THEOREM

For $\varphi : \frac{\mathbb{Z}+c}{a} \to \mathbb{R}$, $\mathcal{E}\left(\left\{\varphi\left(\frac{k+\alpha}{a}\right)\right\}\varphi\right)$ is a Riesz basis for $L^2[0,a]$ if $\sup_{k\in\mathbb{Z}} \left|\frac{k+\alpha}{a} - \varphi\left(\frac{k+\alpha}{a}\right)\right| < \frac{1}{4a}$.

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WEYL EQUIDISTRIBUTION THEOREM

For α irrational it holds

$$\lim_{R\to\infty}\frac{1}{R}\sum_{k=1}^{R} k\,\alpha \mod 1 \;= \frac{1}{2}$$

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Beatty sequences

For a, b irrational with $a + b = 1$, the sets $\mathcal{A} = \left\{\left\lfloor \frac{k}{a} \right\rfloor\right\}_{k\in\mathbb{N}}$ and $\mathcal{B} = \left\{\left\lfloor \frac{k}{b} \right\rfloor\right\}_{k\in\mathbb{N}}$ partition $\mathbb{N}$.

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Proof. There exist $\lfloor aN \rfloor$ elements from $\mathcal{A}$ in $[0,N)$ and $\lfloor bN \rfloor$ elements from $\mathcal{B}$ in $[0,N)$.

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$a, b \notin \mathbb{Q}$ implies

$$aN - 1 < \lfloor aN\rfloor < aN, \quad \text{and} \quad bN - 1 < \lfloor bN\rfloor < bN \quad \text{and in sum}$$

$$N - 2 = aN - 1 + bN - 1 < \lfloor aN\rfloor + \lfloor bN\rfloor < aN + bN = N.$$

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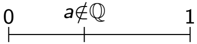
So both sequences together drop 1 element in $[0, 2)$, so onto 1, an additional integer into $[0, 3)$, so onto 2, ...

Proof



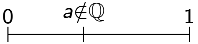
AVDONIN'S THEOREM (Kadec $\frac{1}{4}$-Theorem for R=1) PROOF

0 ⊢—— $a \notin \mathbb{Q}$ ——⊣ 1

For $\varphi : \frac{\mathbb{Z}+\alpha}{a} \to \mathbb{R}$ injective with separated range, $\mathcal{E}\{\varphi(\frac{k+\alpha}{a})\}$ is a Riesz basis for $L^2[0, a]$ if for $R > 0$,

$$\frac{1}{4a} > \sup_{m\in\mathbb{Z}} \left| \frac{1}{R} \sum_{k\in[mR,(m+1)R)} \frac{k+\alpha}{a} - \varphi\left(\frac{k+\alpha}{a}\right) \right|.$$

Avdonin's Theorem (Kadec $\frac{1}{4}$-Theorem for $R=1$) **Proof**



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Weyl-Khinchin Equidistribution Theorem

For a irrational and $\epsilon > 0$ exists R so that for all $m \in \mathbb{Z}$,

$$\epsilon > \left| \frac{1}{R} \sum_{k=mR}^{(m+1)R-1} \frac{k+\frac{1}{2}}{a} \mod 1 \; - \frac{1}{2} \right|$$

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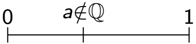

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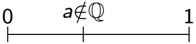
Avdonin's Theorem (Kadec $\frac{1}{4}$-Theorem for $R=1$) **Proof** 

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Seip's Theorem

There exists $\Lambda \subseteq \mathbb{Z}$ such that $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2[0,a]$, $0 < a \leq 1$.

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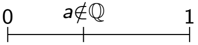

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Inhomogeneous Beatty sequences

For a irrational, the sets $\left\{ \left[\frac{k+\frac{1}{2}}{a} \right]_{\mathbb{Z}+\frac{1}{2}} \right\}_{k\in\mathbb{Z}}$ and $\left\{ \left[\frac{\ell+\frac{1}{2}}{1-a} \right]_{\mathbb{Z}+\frac{1}{2}} \right\}_{\ell\in\mathbb{Z}}$ partition $\mathbb{Z}+\frac{1}{2}$.

Two intervals

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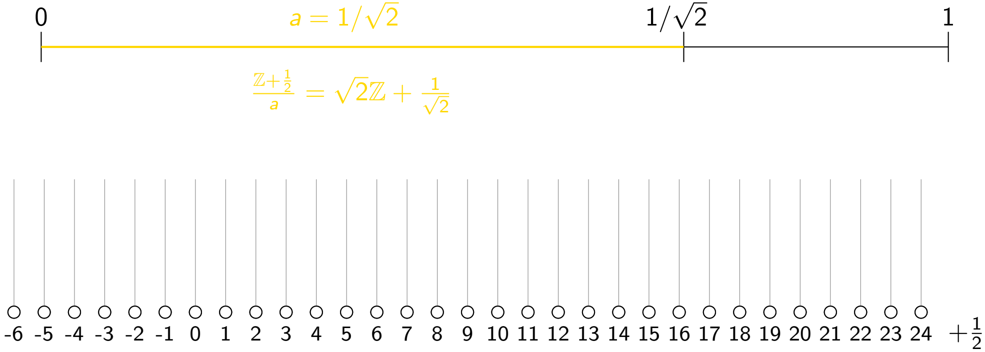
$$\frac{\mathbb{Z}+\frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}$$

Two intervals



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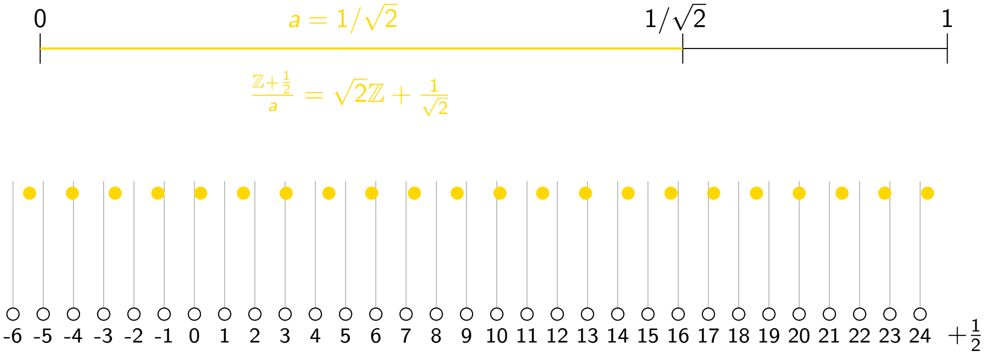

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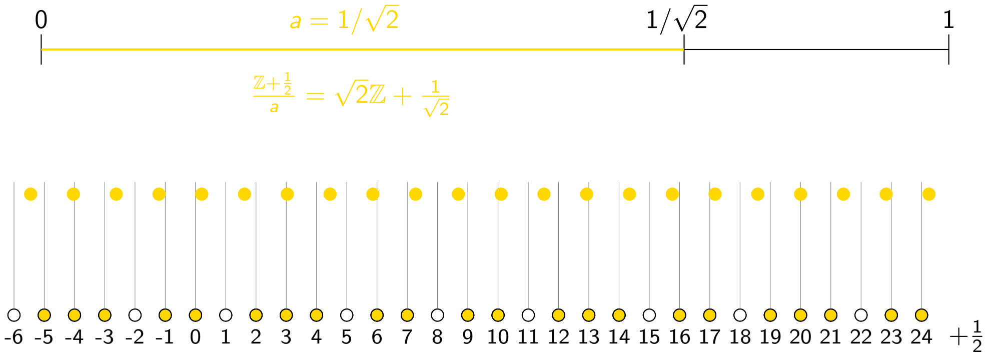

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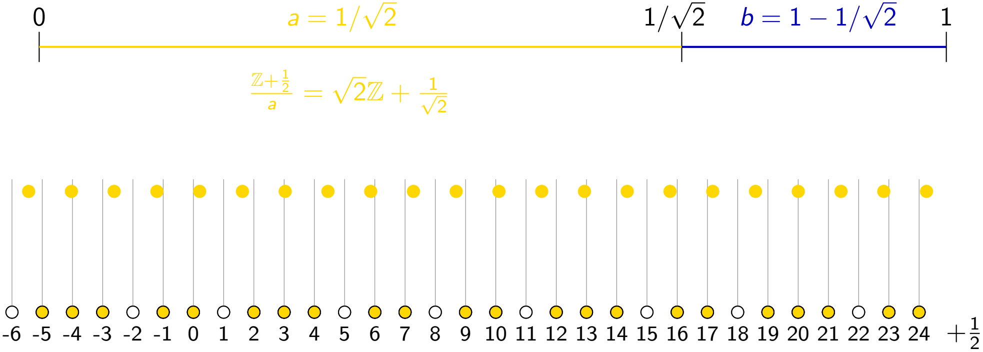

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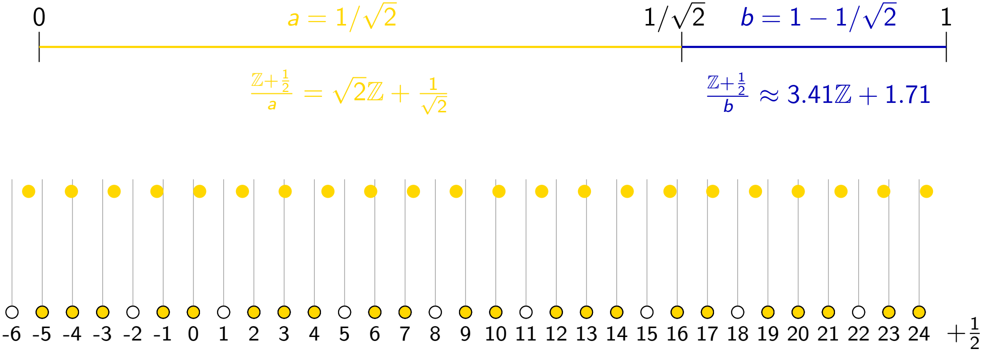

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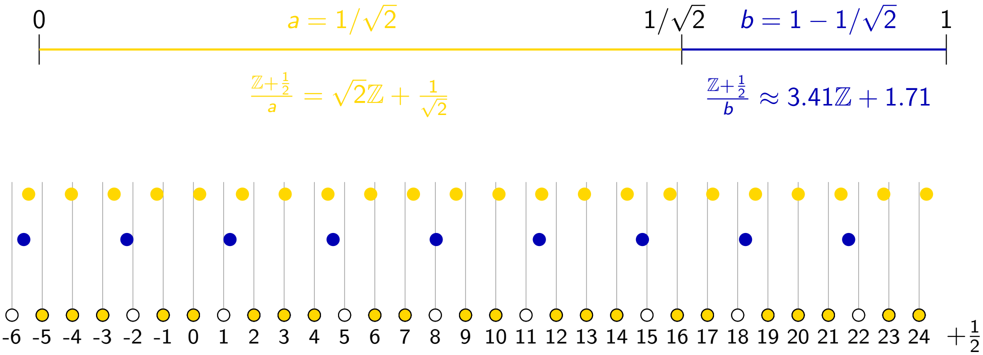

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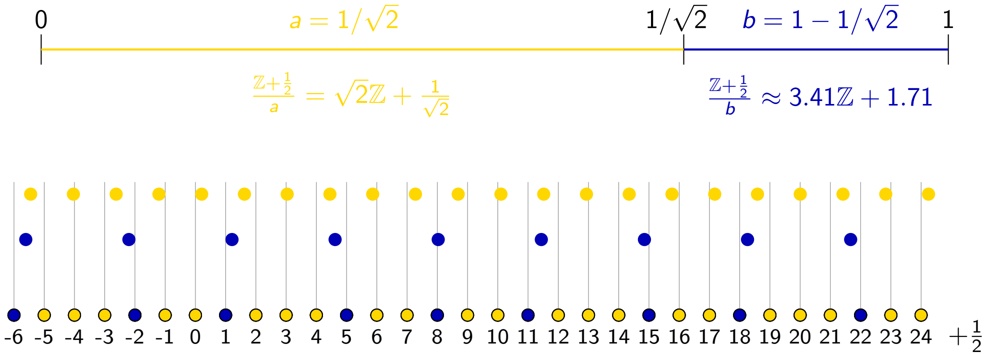

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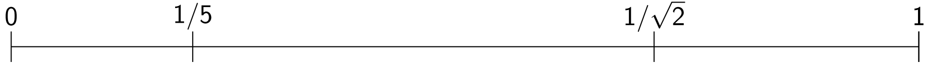

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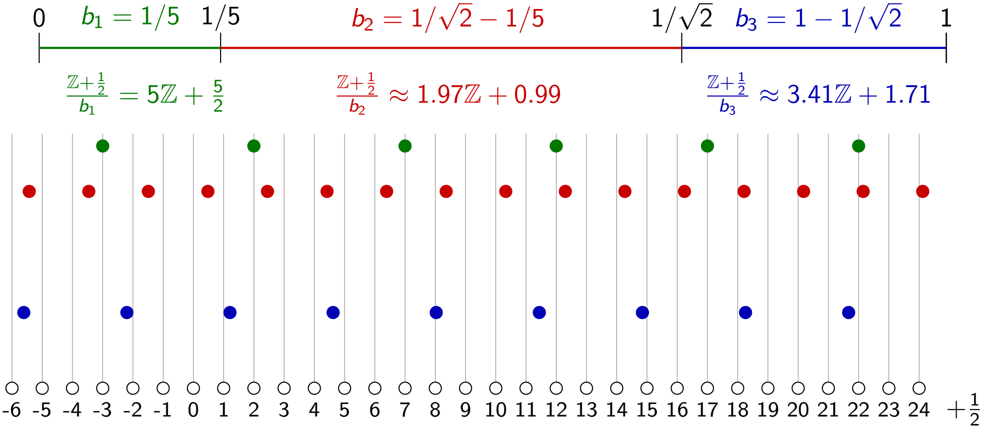

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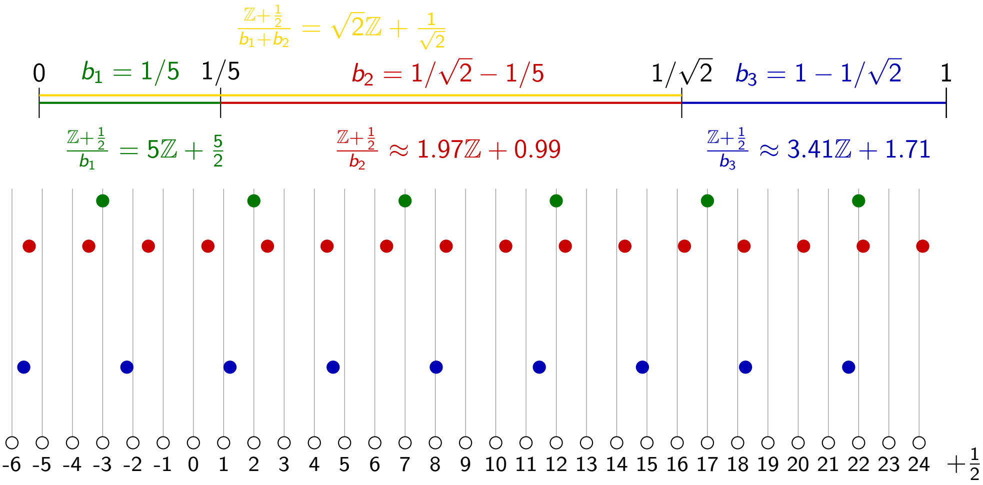

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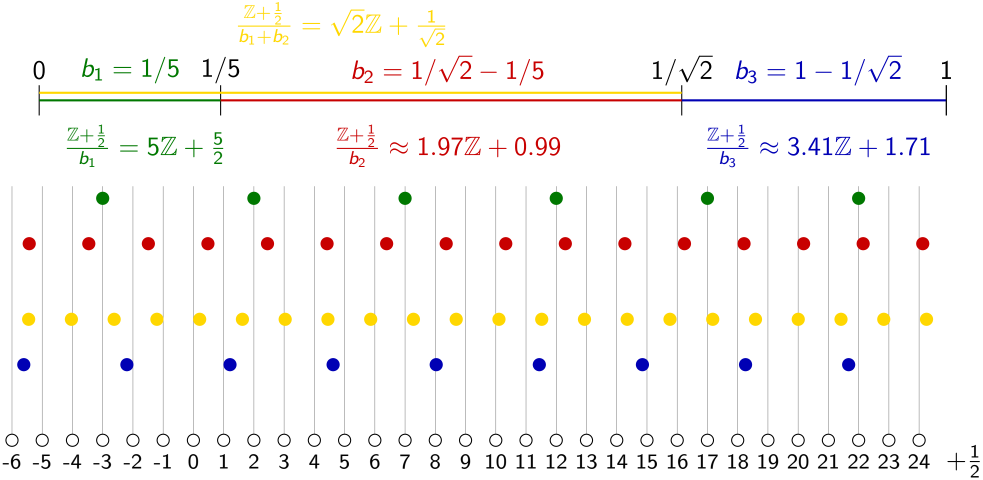

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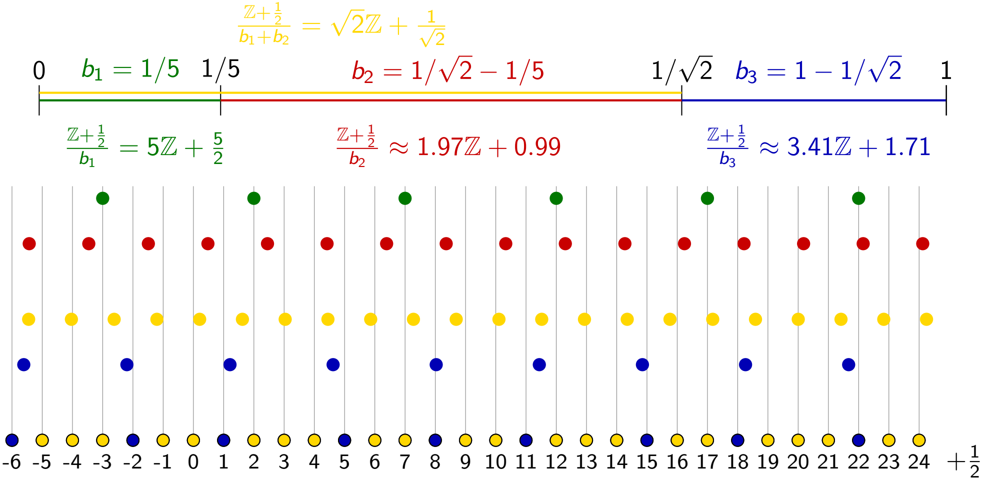
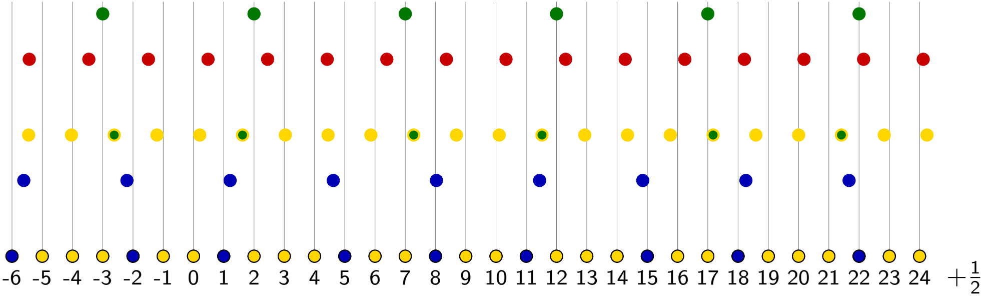


Three intervals

$$\frac{\mathbb{Z}+\frac{1}{2}}{b_1+b_2} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}$$

0 $\quad b_1 = 1/5 \quad$ $1/5$ $\quad b_2 = 1/\sqrt{2} - 1/5 \quad$ $1/\sqrt{2}$ $\quad b_3 = 1 - 1/\sqrt{2} \quad$ 1

$$\frac{\mathbb{Z}+\frac{1}{2}}{b_1} = 5\mathbb{Z} + \frac{5}{2} \qquad \frac{\mathbb{Z}+\frac{1}{2}}{b_2} \approx 1.97\mathbb{Z} + 0.99 \qquad \frac{\mathbb{Z}+\frac{1}{2}}{b_3} \approx 3.41\mathbb{Z} + 1.71$$



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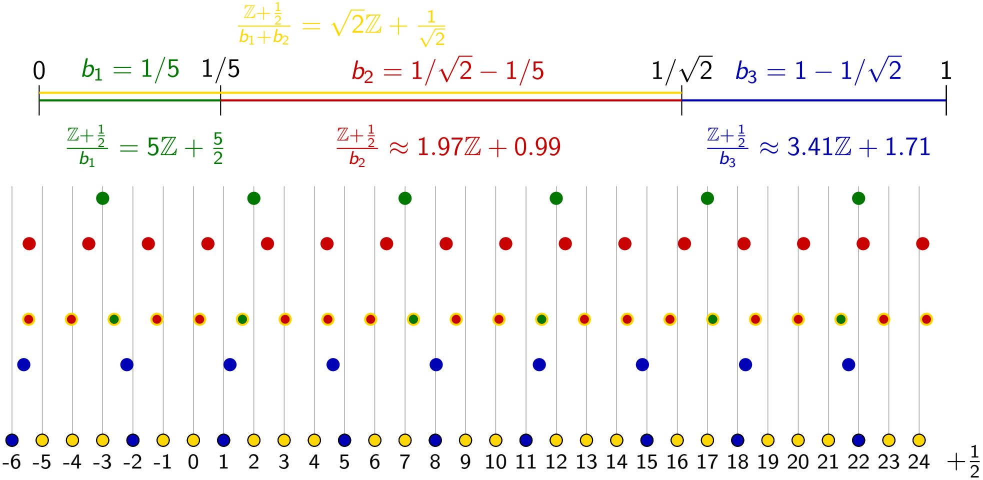

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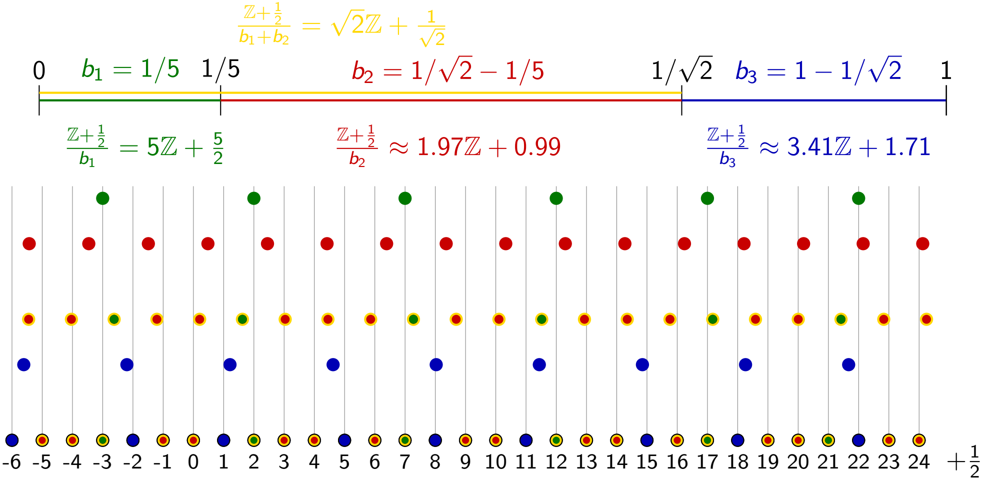

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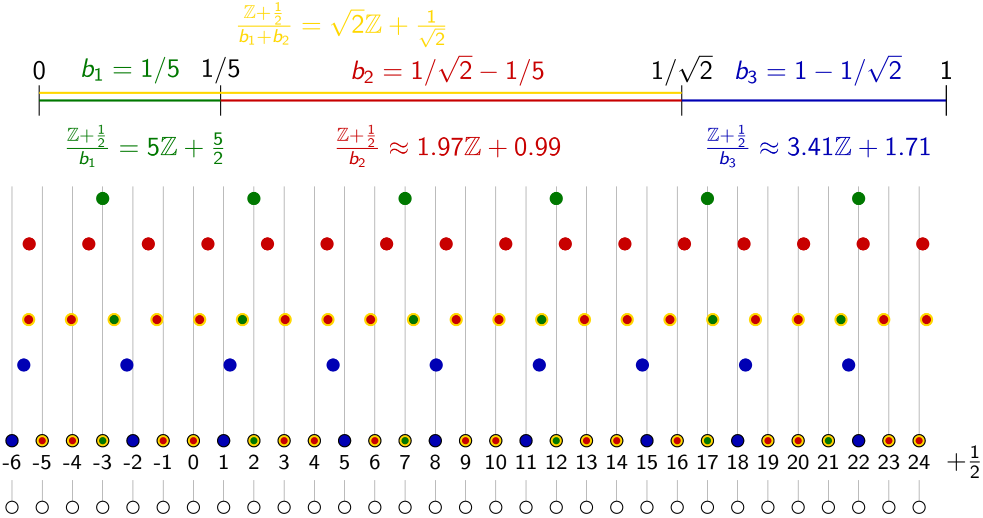

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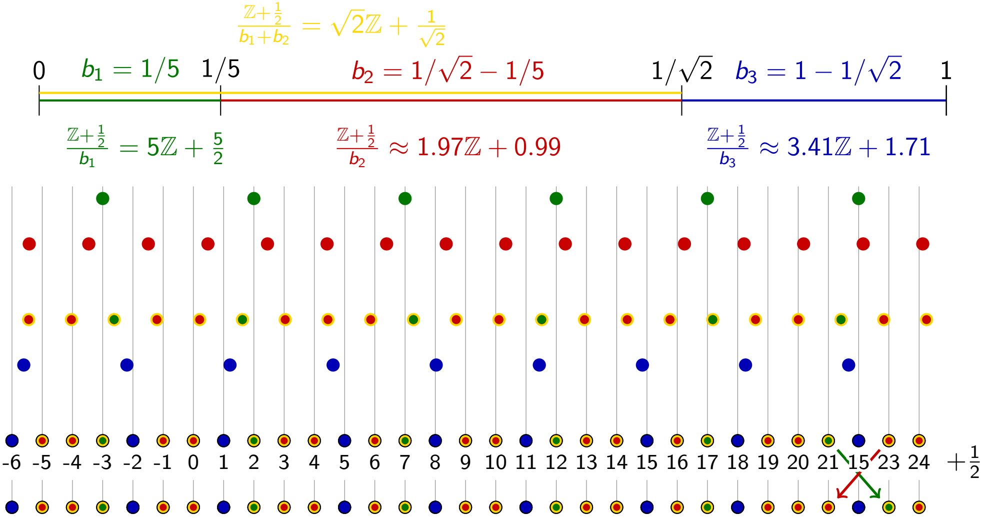

THEOREM B implies

$\mathbb{Z}$ partitions into $\Lambda_1, \ldots, \Lambda_n$ with $\mathcal{E}\left(\bigcup_{r=k}^{\ell} \Lambda_r\right)$ is a Riesz basis of $L^2[a_{k-1}, a_\ell] = L^2\left[\bigcup_{r=k}^{\ell} [a_{k-1}, a_k]\right]$ for all k, ℓ.

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A family of Riesz bases $\mathcal{E}(\Lambda_j)$ of $L^2(S_j)$ is called hierarchical if

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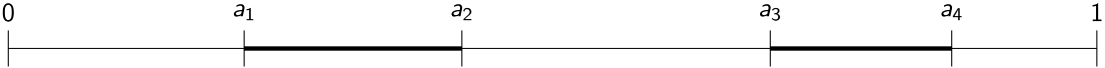

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THEOREM (Kozman, Nitzan 2015)
For any finite union of intervals $S\subseteq[0,1]$ exists $\Lambda\subset\mathbb{Z}$ such that $E(\Lambda)$ is a Riesz basis for $L^2(S)$.

Theorem (Caragea, Lee, 2022)
If $c_1, d_1, \ldots, c_n, d_n$ are linearly independent over $\mathbb{Q}$, then exist $\Lambda_1,...,\Lambda_n \subseteq \mathbb{Z}$ with $\mathcal{E}(\bigcup_{j\in J}\Lambda_j)$ is Riesz of

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THEOREM (Caragea, Lee, Malikiosis, GP)
Let $N = pq$ for p, q distinct primes, $q \le 7$, and $c_1, d_1, \ldots, c_n, d_n \in \{0, \frac{1}{N}, \ldots, \frac{N-1}{N}, 1\}$, then the conclusion above holds.

Proof idea of the rational result

For $N \in \mathbb{N}$ set

$$\mathcal{F}_N = \left(e^{2\pi i \frac{k\ell}{N}}\right)_{0 \le k,\ell \le N-1} = \left(\omega^{k\ell}\right)_{0 \le k,\ell \le N-1}, \quad \omega = e^{-2\pi i \frac{1}{N}}$$

For $K, L \subset \{0, 1, \dots, N-1\}$ of equal cardinality

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Chebotarëv's theorem (1920s)

If N is prime, then, for any K, L of equal cardinality, $\det \mathcal{F}_N[K, L] \neq 0$.

N NOT PRIME

$$\mathcal{F}_4 = \begin{pmatrix} \color{red}{1} & 1 & \color{red}{1} & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ \color{red}{1} & \omega^2 & \color{red}{1} & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega \end{pmatrix} \text{ hence } \det F_4\left[\{0,2\},\{0,2\}\right] = 0$$

$$\mathcal{F}_6 = \begin{pmatrix} \color{red}{1} & 1 & 1 & \color{red}{1} & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ \color{red}{1} & \omega^2 & \omega^4 & \color{red}{1} & \omega^2 & \omega^4 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix} \text{ hence } \det F_6\left[\{0,2\},\{0,3\}\right] = 0$$

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Conjectures

- If N is square-free, then all principal minors of $\mathcal{F}_N$ are non-zero. (Cabrelli, Molter, Negreira; Caragea, Lee, Malikiosis, GP)
- If N contains no fourth power, then exists a permutation σ so that $\det \mathcal{F}_N[K, \sigma(K)] \neq 0$ for all K.

Testing the conjecture

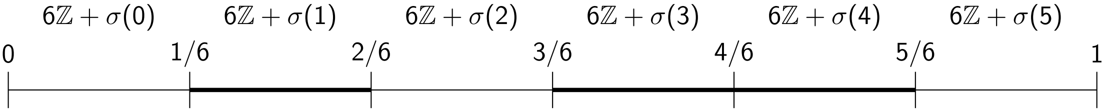
Square-free case

- Numerical and symbolical testing for small N (up to 45).
- Conjecture confirmed for square free $2p, 3p, 5p, 6p, 7p$ and qp for $q > \Gamma_p$ which is large. (Loukaki; Caragea,Lee, Malikiosis, GP; Emmerich, Kunis)
- Smallest open cases $11 \cdot 13 = 143$, $2 \cdot 5 \cdot 7 = 70$, $2 \cdot 3 \cdot 5 \cdot 7 = 210$.

Non square-free case

- Initial numerical and symbolical testing for small N.
- No 'good' permutations seem to exist for $N = 16 = 2^4$.
- Next interesting case $3^4 = 81$.

Back to hierarchical Riesz bases

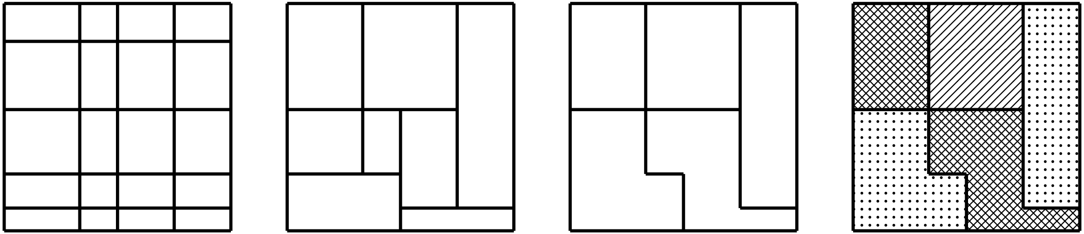


If $\det \mathcal{F}_N[K, \sigma(K)] \neq 0$ for all K then the family of Riesz bases $\mathcal{E}(N\mathbb{Z}+\sigma(k))$ of $L^2([\frac{k}{N}, \frac{k+1}{N}]$ is hierarchical.

Recall: σ=identity works for square free $2p, 3p, 5p, 6p, 7p$ and qp for $q > \Gamma_p$ which is large. (Loukaki; Caragea,Lee, Malikiosis, GP; Emmerich, Kunis)

$$d = 2$$

Generalizations $d = 2$



$$d \geq 2$$

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Question rephrased

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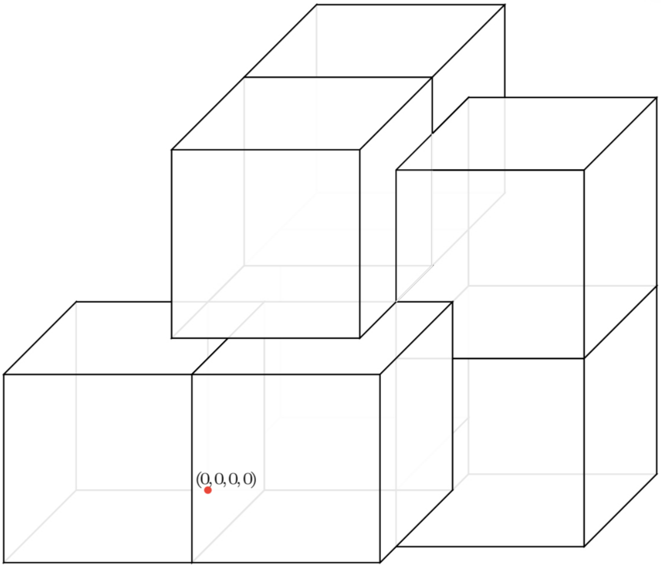
(0,0,0,0)

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THEOREM (Kolountzakis 24^+)
$\Phi \subseteq A\mathbb{Z}^d$ with $([0,1)^d, \Phi)$ is a tiling implies $\det A = 1/N$.

Why 7?

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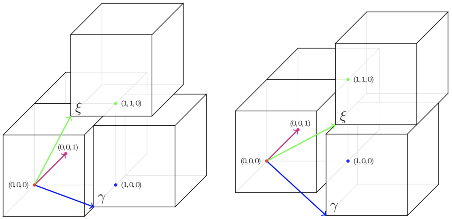

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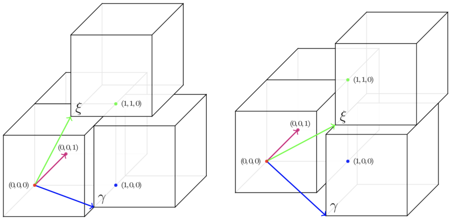

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Every lattice Λ in $\mathbb{R}^d$ with $([0,1)^d, \Lambda)$ is a cube tiling has a twin pair.

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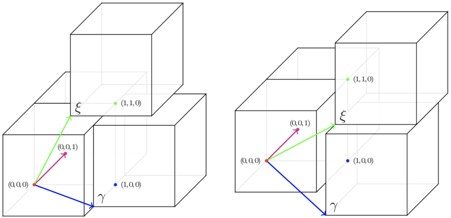

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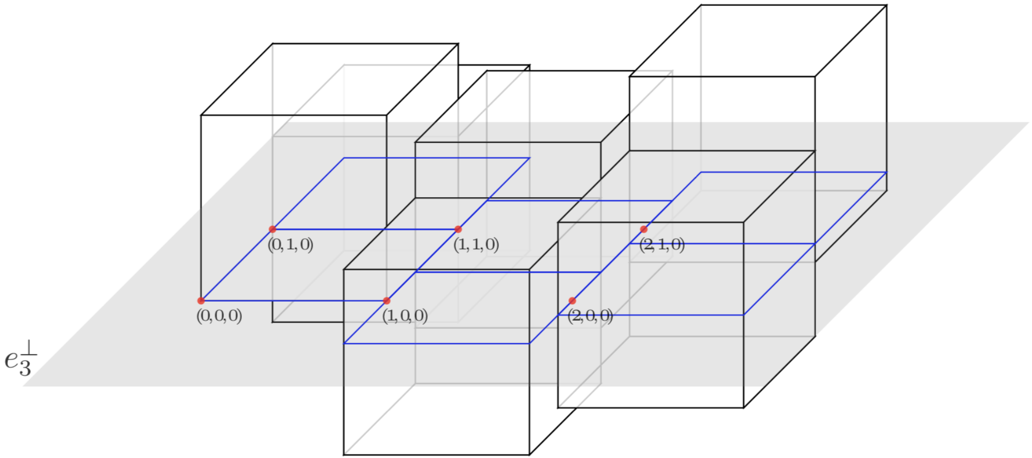
Theorem (Keller conjectured 1930; Perron confirmed for $d \leq 6$, 1940; Lagarias & Shor gave counterexample for $d = 10$, 1992; last open case $d = 7$ settled by computer algebra in 2020)

Every set Φ in $\mathbb{R}^d$ with $([0,1)^d, \Phi)$ is a cube tiling has a twin pair if and only if $d \leq 7$.


(0, 1, 0)
(1, 1, 0)
(2, 1, 0)
(0, 0, 0)
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$e_3^\perp$